Poisson structures compatible with the canonical metric of $\mathbb{R}^3$

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Abstract. In this Note, we will characterize the Poisson structures compatible with the canonical metric of $\mathbb{R}^3$. We will also give some relevant examples of such structures. The notion of compatibility between a Poisson structure and a Riemannian metric used in this Note was introduced and studied by the author in [1], [2], [3].

1 Introduction and main results

Many fundamental definitions and results about Poisson manifolds can be found in Vaisman's monograph [5].

As a continuation of the study by the author of Poisson structures compatible with Riemannian metrics in [1], [2] and [3], it is of interest to find some relevant examples of such structures in the low dimensions. So we were interested in finding all the Poisson structures compatible with the canonical metric in $\mathbb{R}^3$. The results of this search is the theme of this Note.

Let us recall some facts about the notion of compatibility between a Poisson structure and a Riemannian metric in order to motivate our investigation and to show the interest of this Note.

Let P be a Poisson manifold with Poisson tensor π. A Riemannian metric on T^*P is a smooth symmetric contravariant 2-form $<,>$ on P such that, at each point $x \in P$, $<,>_x$ is a scalar product on T^*_xP. For each Riemannian metric $<,>$ on T^*P, we consider the contravariant connection D introduced in [1] by

$$\begin{aligned} 2 < D_\alpha\beta,\gamma > &= \pi(\alpha).<\beta,\gamma> + \pi(\beta).<\alpha,\gamma> - \pi(\gamma).<\alpha,\beta> \\ &\quad + <[\alpha,\beta]_\pi,\gamma> + <[\gamma,\alpha]_\pi,\beta> + <[\gamma,\beta]_\pi,\alpha>, \end{aligned} \tag{1}$$

where $\alpha, \beta, \gamma \in \Omega^1(P)$ and the Lie bracket $[\ ,\ ]_\pi$ is given by

$$[\alpha, \beta]_\pi = L_{\pi(\alpha)}\beta - L_{\pi(\beta)}\alpha - d(\pi(\alpha, \beta));$$

here, $\pi : T^*P \longrightarrow TP$ denotes the bundle map given by

$$\beta[\pi(\alpha)] = \pi(\alpha, \beta).$$

The connection D is the contravariant analogue of the usual Levi-Civita connection. The connection D has vanishing torsion, i.e.

$$D_\alpha\beta - D_\beta\alpha = [\alpha, \beta]_\pi.$$

Moreover, it is compatible with the Riemannian metric $<,>$, i.e.

$$\pi(\alpha). < \beta, \gamma > = < D_\alpha\beta, \gamma > + < \beta, D_\alpha\gamma > .$$

The notion of contravariant connection has been introduced by Vaisman (see [5] p.55) as contravariant derivative. Recently, a geometric approach of this notion was given by Fernandes in [4].
If we put, for any $f \in C^\infty(P)$,

$$\phi_{<,>}(f) = \sum_{i=1}^{n} < D_{\alpha_i}df, \alpha_i > \tag{2}$$

where $(\alpha_1, \ldots, \alpha_n)$ is a local othonormal basis of 1-forms, we get a derivation on $C^\infty(P)$ and hence a vector field called the modular vector field of (P, π) with respect to the metric $<,>$.
The couple $(\pi, <,>)$ is compatible if, for any $\alpha, \beta, \gamma \in \Omega^1(P)$,

$$D\pi(\alpha, \beta, \gamma) := \pi(\alpha).\pi(\beta, \gamma) - \pi(D_\alpha\beta, \gamma) - \pi(\beta, D_\alpha\gamma) = 0. \tag{3}$$

In this case, the triple $(P, \pi, <,>)$ is called a Riemann-Poisson manifold.
Riemann-Poisson manifolds was first introduced by the author in [1]. Let us summarize some important results of Riemann-Poisson manifolds proved by the author in [2] and [3].
For a Riemann-Poisson manifold $(P, \pi, <,>)$ the following results are true:
1. the symplectic leaves are Kählerian;
2. the symplectic foliation (when it is a regular foliation) is a Riemannian foliation;

3. (P, π) is unimodular (see [6] for the details on the notion of unimodular Poisson manifolds) and moreover the modular vector field $\phi_{<,>}$ given by (2) vanishes.
With those properties in mind, we can give the main results of this Note.

Theorem 1.1 *A Poisson tensor* $\pi = \pi_{12}\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + \pi_{13}\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} + \pi_{23}\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}$ *is compatible with the canonical metric* $<,>$ *of* $\mathbb{R}^3$ *iff there exists a function* $f \in C^\infty(\mathbb{R}^3)$ *such that*

$$\pi_{12} = \frac{\partial f}{\partial z}, \qquad \pi_{13} = -\frac{\partial f}{\partial y}, \qquad \pi_{23} = \frac{\partial f}{\partial x},$$

and

$$d(< df, df >) - \Delta(f)df = 0 \tag{E}$$

where $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ *is the usual Laplacian on* $\mathbb{R}^3$*. Moreover, the function* f *is a Casimir function of* π*.*

The following proposition and the theorem above give all the linear Poisson structures on $\mathbb{R}^3$ compatible with the canonical metric.

Proposition 1.1 *The polynomial functions of degree 2 solutions of* (E) *are*

$$f(x, y, z) = (a + c)x^2 + (a + b)y^2 + (b + c)z^2 - 2\sqrt{bc}xy + 2\sqrt{ab}xz + 2\sqrt{ac}yz,$$

where $a, b, c \in \mathbb{R}$ *and* $ab, ac, bc \in \mathbb{R}_+$*.*

Let $\pi_{so(3)} = z\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} + x\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}$ be the linear Poisson structure on $\mathbb{R}^3$ corresponding to the Lie algebra $so(3)$. In [2], we have shown that there isn't any Riemannian metric on $\mathbb{R}^3$ compatible with $\pi_{so(3)}$. However, we have the following proposition.

Proposition 1.2 *The function* $f(x, y, z) = (x^2 + y^2 + z^2)^{\frac{3}{2}}$ *is a solution of* (E) *and then* $(x^2 + y^2 + z^2)^{\frac{1}{2}}\pi_{so(3)}$ *is compatible with the canonical metric of* $\mathbb{R}^3$*.*

Remarks. 1. The fact that there isn't any metric compatible with $\pi_{so(3)}$ and, however, the Poisson structure $(x^2 + y^2 + z^2)^{\frac{1}{2}}\pi_{so(3)}$ is compatible with the canonical metric seems curious. But it can be explained easily. In fact, let

$(P,\pi,<,>)$ be a Poisson manifold with a contravariant Riemannian metric. If we change the Poisson structure by $f\pi$ where $f\in C^\infty(P)$, the contravariant Levi-Civita connection given by (1) become more complicated and is given by

$$D^{f\pi}_\alpha\beta = fD^\pi_\alpha\beta + \frac{1}{2}\pi(\alpha,\beta)df - \frac{1}{2} < df,\beta > J\alpha - \frac{1}{2} < df,\alpha > J\beta$$

where J is the field of homomorphisms given by $\pi(\alpha,\beta) =< J\alpha,\beta >$.
2. The Poisson structures $\pi_{so(3)}$ and $(x^2+y^2+z^2)^{\frac{1}{2}}\pi_{so(3)}$ have the same symplectic foliation and, in restriction to a symplectic leaf, the two symplectic structures differ by a constant.
3. It is possible that the compatibility of $(x^2+y^2+z^2)^{\frac{1}{2}}\pi_{so(3)}$ with the canonical metric has some physical signification.

2 Proof of Theorem 1.1

Let π be a bivectors field on $\mathbb{R}^3$ given by

$$\pi = \pi_{12}\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y} + \pi_{13}\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z} + \pi_{23}\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}.$$

We consider the canonical metric $<,>$ on $\mathbb{R}^3$ as contravariant metric given by

$$< dx,dx >=< dy,dy >=< dz,dz >= 1, < dx,dy >=< dx,dz >=< dy,dz >= 0.$$

We denote by D the Levi-Civita contravariant connection associated with $(\pi,<,>)$.
Firstly, remark that the compatibility between π and $<,>$ implies the vanishing of the modular vector field given by

$$\phi_{<,>}(f) =< D_{dx}df,dx > + < D_{dy}df,dy > + < D_{dz}df,dz > .$$

A straightforward calculation shows that the vanishing of $\phi_{<,>}$ is equivalent to

$$\frac{\partial\pi_{12}}{\partial y} + \frac{\partial\pi_{13}}{\partial z} = 0, \qquad \frac{\partial\pi_{12}}{\partial x} - \frac{\partial\pi_{23}}{\partial z} = 0, \qquad \frac{\partial\pi_{13}}{\partial x} + \frac{\partial\pi_{23}}{\partial y} = 0. \qquad (4)$$

Now, it is easy to see that (4) is equivalent to the fact that $\pi_{23}dx - \pi_{13}dy + \pi_{12}dz$ is a closed 1-form and hence is exact. So, there exists a function $f \in C^\infty(\mathbb{R}^3)$ such that

$$\pi_{12} = \frac{\partial f}{\partial z}, \qquad \pi_{13} = -\frac{\partial f}{\partial y}, \qquad \pi_{23} = \frac{\partial f}{\partial x}. \tag{5}$$

Now, let us compute the contravariant connection D. We will use the Christoffel symbols Γ_{ij}^k. For example, $D_{dx}dx = \Gamma_{11}^1 dx + \Gamma_{11}^2 dy + \Gamma_{11}^3 dz$. From (1), we get:

$$\begin{aligned}
&\Gamma_{11}^1 = 0, \quad \Gamma_{11}^2 = -\frac{\partial^2 f}{\partial x \partial z}, \quad \Gamma_{11}^3 = \frac{\partial^2 f}{\partial x \partial y},\\
&\Gamma_{12}^1 = \frac{\partial^2 f}{\partial x \partial z}, \quad \Gamma_{12}^2 = 0, \quad \Gamma_{12}^3 = \frac{1}{2}\left(-\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right),\\
&\Gamma_{21}^1 = 0, \quad \Gamma_{21}^2 = -\frac{\partial^2 f}{\partial y \partial z}, \quad \Gamma_{21}^3 = \frac{1}{2}\left(-\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial z^2}\right),\\
&\Gamma_{13}^1 = -\frac{\partial^2 f}{\partial x \partial y}, \quad \Gamma_{13}^2 = \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial z^2}\right), \quad \Gamma_{13}^3 = 0,\\
&\Gamma_{31}^1 = 0, \quad \Gamma_{31}^2 = \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial z^2}\right), \quad \Gamma_{31}^3 = \frac{\partial^2 f}{\partial y \partial z},\\
&\Gamma_{22}^1 = \frac{\partial^2 f}{\partial y \partial z}, \quad \Gamma_{22}^2 = 0, \quad \Gamma_{22}^3 = -\frac{\partial^2 f}{\partial x \partial y},\\
&\Gamma_{23}^1 = \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right), \quad \Gamma_{23}^2 = \frac{\partial^2 f}{\partial x \partial y}, \quad \Gamma_{23}^3 = 0,\\
&\Gamma_{32}^1 = \frac{1}{2}\left(-\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right), \quad \Gamma_{32}^2 = 0, \quad \Gamma_{32}^3 = -\frac{\partial^2 f}{\partial x \partial z},\\
&\Gamma_{33}^1 = -\frac{\partial^2 f}{\partial y \partial z}, \quad \Gamma_{33}^2 = \frac{\partial^2 f}{\partial x \partial z}, \quad \Gamma_{33}^3 = 0.
\end{aligned}$$

Now, we will compute $D_{dx}\pi$, $D_{dy}\pi$ and $D_{dz}\pi$. We have

$$\begin{aligned}
D_{dx}\pi &= \pi(dx)(\frac{\partial f}{\partial z})\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} - \pi(dx)(\frac{\partial f}{\partial y})\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} + \pi(dx)(\frac{\partial f}{\partial x})\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}\\
&+ \frac{\partial f}{\partial z}\left((D_{dx}\frac{\partial}{\partial x}) \wedge \frac{\partial}{\partial y} + \frac{\partial}{\partial x} \wedge (D_{dx}\frac{\partial}{\partial y})\right)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\partial f}{\partial y}\left((D_{dx}\frac{\partial}{\partial x})\wedge\frac{\partial}{\partial z}+\frac{\partial}{\partial x}\wedge(D_{dx}\frac{\partial}{\partial z})\right)\\
&+ \frac{\partial f}{\partial x}\left((D_{dx}\frac{\partial}{\partial y})\wedge\frac{\partial}{\partial z}+\frac{\partial}{\partial y}\wedge(D_{dx}\frac{\partial}{\partial z})\right).
\end{aligned}
$$

On other hand, we have

$$
\begin{aligned}
\pi(dx) &= \frac{\partial f}{\partial z}\frac{\partial}{\partial y}-\frac{\partial f}{\partial y}\frac{\partial}{\partial z},\\
D_{dx}\frac{\partial}{\partial x} &= -\frac{\partial^2 f}{\partial x\partial z}\frac{\partial}{\partial y}+\frac{\partial^2 f}{\partial x\partial y}\frac{\partial}{\partial z},\\
D_{dx}\frac{\partial}{\partial y} &= \frac{\partial^2 f}{\partial x\partial z}\frac{\partial}{\partial x}+\frac{1}{2}\left(-\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}\right)\frac{\partial}{\partial z},\\
D_{dx}\frac{\partial}{\partial z} &= -\frac{\partial^2 f}{\partial x\partial y}\frac{\partial}{\partial x}+\frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}-\frac{\partial^2 f}{\partial y^2}-\frac{\partial^2 f}{\partial z^2}\right)\frac{\partial}{\partial y}.
\end{aligned}
$$

Substituting those expressions into the expression of $D_{dx}\pi$, we get

$$
\begin{aligned}
D_{dx}\pi &= \left(\frac{\partial f}{\partial z}\frac{\partial^2 f}{\partial y\partial z}+\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x\partial y}+\frac{1}{2}\frac{\partial f}{\partial y}\left(-\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}-\frac{\partial^2 f}{\partial z^2}\right)\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}\\
&+ \left(\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial y\partial z}+\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x\partial z}+\frac{1}{2}\frac{\partial f}{\partial z}\left(-\frac{\partial^2 f}{\partial x^2}-\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}\right)\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z}.
\end{aligned}
$$

In the same manner we can get

$$
\begin{aligned}
D_{dy}\pi &= \left(-\frac{\partial f}{\partial z}\frac{\partial^2 f}{\partial x\partial z}-\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial x\partial y}+\frac{1}{2}\frac{\partial f}{\partial x}\left(-\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}\right)\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}\\
&+ \left(\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial y\partial z}+\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x\partial z}+\frac{1}{2}\frac{\partial f}{\partial z}\left(-\frac{\partial^2 f}{\partial x^2}-\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}\right)\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}.\\
D_{dz}\pi &= \left(-\frac{\partial f}{\partial z}\frac{\partial^2 f}{\partial x\partial z}-\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial x\partial y}+\frac{1}{2}\frac{\partial f}{\partial x}\left(-\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}\right)\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z}\\
&+ \left(-\frac{\partial f}{\partial z}\frac{\partial^2 f}{\partial y\partial z}-\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x\partial y}+\frac{1}{2}\frac{\partial f}{\partial y}\left(+\frac{\partial^2 f}{\partial x^2}-\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}\right)\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}.
\end{aligned}
$$

Now, it is easy to show that $D\pi = 0$ iff f satisfies (E). It is also easy to show that f is a Casimir function. Remark that $D\pi = 0$ implies that the bracket of Schouten $[\pi, \pi]$ vanishes which finish the proof of Theorem 1.1.□

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